A CASE STUDY OF CONFLICTING REALIZATIONS OF CONTINUITY

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In this paper I present a case study to illustrate conflicts between different 'realizations' of the concept of 'continuous function' held by a university first year student. Sfard's commognitve framework is used in the analysis of a student's work on continuity. I point out how these conflicting realizations have arisen from the inconsistent definitions presented in text books and other mathematical resources. The study also points to the need of extending the notion of "commognitive conflict" in the framework.

BACKGROUND AND THEORETICAL FRAMEWORK

This paper reports on a phenomenon identified in the second stage of data analysis of a larger study. The study is guided by the research question "what are the effects of different definitions of 'continuity of a function' on student learning?" The concept of 'continuity' has been recognized as a difficult topic in Calculus and many studies have been done on student understanding of the concept and how students make relations between continuity and other Calculus topics (e.g.: Bezuidenhout, 2001; Vinner, 1987; Cornu, 1991; Aspinwall et al., 1997). In an analysis of text books and other mathematical resources that was carried out as part of the current study, two issues were identified pertaining to definitions of continuity that are inconsistent with each other. These two problematic situations are described in the succeeding section. It was found in the first stage of the current study that university first year students have difficulties in determining whether a function is continuous or not when the function is not defined on an interval in particular. What is reported in this paper is the tension displayed in the discourse of a particular participant when she was trying to determine whether a particular function was continuous or not. I discuss how these tensions or rather 'conflicts' are arising from the inconsistent definitions of continuity.

The familiar notion of 'cognitive conflict' has been attended to by many mathematics education researchers (e.g.: Zazkis & Chernoff, 2006; Tall, 1977; Tirosh & Graeber, 1990). The notion has relations to Piaget's equilibration theory, Festinger's theory of cognitive dissonance, and Berlyne's theory of conceptual conflict (Stylianides & Stylianides, 2008). A cognitive conflict is said to be "invoked when a learner is faced with contradiction or inconsistency in his or her ideas" (Zazkis & Chernoff, 2008, p. 196).

However, with the new directions taken in looking at 'thinking' in the recent years from cognitive theories towards discursive theories, my study is informed by Sfard's commognitive theory and its interpretation of 'conflict'.

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Sfard (2008) unifies thinking and communication as commognition. In the commognitive framework, thinking is conceptualized as an individualized version of interpersonal communication. With the visioning of Mathematics as a discourse, it is claimed to be an autopoetic system that creates the objects of its study. Hence mathematical objects are discursive objects and students personally construct these mathematical objects which can be represented as 'realization trees'. A realization tree shows the different realizations of a particular signifier where a signifier is a word or symbol that acts as a noun in the mathematical discourse. A realization is a perceptually accessible thing so that narratives about the signifier can be translated into narratives about the realization. Sfard coins "commognitive conflict" as "the encounter between interlocutors who use the same signifiers (words or written symbols) in different ways or perform the same mathematical task according to differing rules" (Sfard, 2008, p. 161). What this paper reports on is different from 'commognitive conflict', in that the conflict is between different realizations (for the same signifier) of the *same* individual.

What follows is a brief introduction to the problems regarding definitions of continuity which has a direct relation to the case study of conflicting realizations.

CONTINUITY: TWO DEFINITIONS

Problem 1: Inconsistent definitions

In the context of an introductory calculus course, and also in many other common resources, the definitions used for continuity related concepts are the limit definitions. There are two different limit definitions (that are labelled as D1 and D2 for reference in this paper) used for "continuity at a point" (and accordingly "discontinuity at a point") on which the other related concepts of continuity can be based on. Below are the two definitions.

D1 (e.g.: Stewart, 2012; Tan, Menz, & Ashlock, 2011)

A function f is said to be continuous at c if,

1. $f(x)$ is defined at $x = c$
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x)$ is equal to $f(c)$

f is discontinuous if any of the above conditions are not satisfied.

D2 (e.g.: Stahl, 2011; Strang, 1991)

A function f is said to be continuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) = f(c)$$

And f is discontinuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) \neq f(c)$$

The deciding factor that makes a definition consistent with either D1 or D2 is the treatment of a point at which the function is not defined. According to D1, a function that is not defined at a point is discontinuous at that point, while according to D2 the question of continuity or discontinuity shouldn't arise.

A 'continuous function' too is defined in two ways where one is in accordance with D1 while the other one is in accordance with D2.

D1 (e.g.: Anton, 1995; Mathematics Harvey Mudd Collage, n.d.)

> A function is a continuous function if it is continuous at every real number.

D2 (e.g.: Strang, 1991; Bogley & Robson, 1996)

> A function is a continuous function if it is continuous in its domain.

Problem 2: Absence of a definition for 'a continuous function'

I have examined several dozen of resources (textbooks, websites, mathematical dictionaries) seeking a definition for a 'continuous function'. In most of the resources such a definition was not explicitly stated. However, the phrase 'continuous function' is loosely used in many places.

The topic of continuity starts off, in many textbooks and websites, with the definition of 'continuity *at a point*' (e.g.: Stewart, 2012). This definition is the leading definition and other related extensions to the concept of continuity of a function, each of which has its own definition may follow (e.g.: continuity on an interval, types of discontinuities, one-sided continuities).

However, the heart of the second problem is that these definitions of continuity/discontinuity at a point are not followed by the definition of a *continuous function* (e.g.: Neuhauser, 2011; Stewart, 2012). This situation leaves room for students, if not explained by the instructor, to intentionally or unintentionally 'construct' a meaning for "continuous function". Instinctively it is likely that this will be interpreted as "continuous everywhere" with 'everywhere' to mean either "all reals" or "domain". Therefore this situation holds the potential to lead students to construct their own meaning for a 'continuous function', which could be in discord with the intended definition.

METHODOLOGY

My data comes from the ongoing research study. The case study I'm presenting is of a first year university student, student 'J', who takes an introductory Calculus course. She was given a questionnaire where she was asked to give the definitions for "continuity at a point" (which she had learnt in the course) and "continuous function" (which she was not taught in the course) and then was given 6 functions in their graphical form to be identified as continuous or discontinuous. Then she was interviewed one on one to discuss her responses.

The first four graphs, which are discussed in the excerpt, are given in Table 1. Note that the domain for graph D was specified.

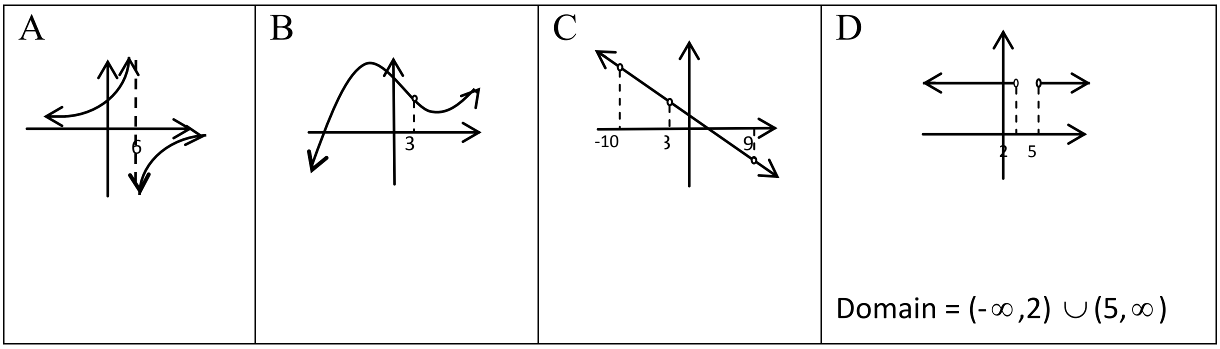

Table 1: The first four graphs in the questionnaire

RESULTS AND ANALYSIS

A realization tree for 'a continuous function' for 'J' was constructed based on her responses to the questionnaire and her utterances in the interview. Among other realizations, it was found that, 'J' had the following two realizations for a continuous function.

X: For every point c in its domain, $f(c)$ is defined and $\lim_{x \to c} f(x) = f(c)$.

[this is in accordance with D2]

Y: A function that does not have holes or asymptotes. [this is in accordance with D1]

Following (Table 2) is an interpretative elaboration ("*interpretative elaboration* is a text that, utterance by utterance, elaborates on the text produced by the interlocutors" (Sfard, 2008, p. 139)) of an excerpt from the interview with 'J' that illustrates the tension between these two realizations. A word that is stressed by an interlocutor is indicated by bold letters. 'G' is the researcher who conducted the interview.

'J' faces this tension when trying to decide the continuity of the graph D that is not defined on an interval.

No.	Who said	What was said	What was done	Interpretative elaboration
118	G	Umm, so here you **refrain** from saying that it is..	Pointing to graph D	'G' is pointing out that even though 'J' has clearly classified graphs A, B and C as "not continuous", she refrained from classifying graph D as "not continuous" but just stating the "discontinuities".
119	G	Here you said no, no, no	Pointing to graphs A, B & C	
120	G	But here you are just saying 'there is a discontinuity'	Pointing back to graph D	
121	J	Yeah		

122	G	At x equals 2 **and** x equals 5		'G' stresses on 'and' because 'J' did not classify the whole interval from 2-5 as a discontinuity but only 2 **and** 5.
123	J	Yeah I wasn't		'J' admits that she avoided this classification in graph D
124	G	Can you explain that to me?		
125	J	I wasn't sure; I did this question for like three minutes…		'J' specifying three minutes for graph D implies that she took *more* time for it than she took to do each of the graphs A, B and C. By saying she took 3 minutes and admitting she wasn't sure of this she's implying that it was challenging to her.
126	G	Ohh		
127	J	Because…and then I went back and looked at the definition and I saw that it was like within the domain that it's given..	Pointing to the definition which is realization A	This is the first graph for which she refers to the definition (X). She did the first three without referring to the definition. And now, she pays attention to the domain because now the domain is "given". And the definition mentions about the domain.
128	G	Hmmm?		
129	J	And then I was like.. oh but there is like a.. open circles…it should be…..		'J' says "but there is like a.. open circles". She uses 'but' because to her, open circles, as in graphs B and C, mean 'discontinuities' (Y) but she's trying to say that they are not in the domain (X). In other words, this utterance could be reworded as "the function should be continuous according to the definition (X) **but** since there are open circles the function has discontinuities (Y)"
130	J	…	Thinking for 2 seconds	The pauses taken to think shows how much she is struggling to decide because there is a battle between two realizations she has for continuity.
131	J	There is a…there is no...umm…		The two utterances "there is a" and "there is no" that take place adjacently clearly indicates the

				conflicting conclusions about continuity of function D resulted through the two different realizations.
132	J	I don't know because it's not con… like within the domain.. it's not a square bracket		'J' wants to say that the function is not continuous ('not con…') but she is stuck because the two points 2 and 5 are not in the domain ('not a square bracket')
133	G	Yeah		
134	J	So it's not…	Pauses	'J' really wants to say it is not continuous and this shows that for her, realization B is stronger than A.
135	G	So		
136	J	I don't really know		'J' is utterly confused and gives up. She doesn't seem to be aware that the confusion stems from two different realizations.

Table 2: Interpretive elaboration for Jennifer's utterances from 118 to 136 that elaborates a conflict between the realizations X and Y for continuity

The tension between the two realizations X and Y which are based on the two inconsistent definitions D1 and D2 is clearly visible in 'J''s utterances. She had learnt D1 as the definition for 'continuity at a point' in her class. She did not have any problem in deciding the continuity of the first three graphs as these were familiar graphs to her that she had often come across in the class. And as she admitted in the interview she did not refer to the definition in deciding whether they were continuous or not. This was an immediate realization (Y) for the signifier 'continuous function' for her that included familiar features that she had seen in functions that were not 'continuous'; holes and asymptotes. The unfamiliarity of the graph D, one with a discontinuity on an interval, pushed her towards the realization X which is the definition she had taken from a website which is consistent with D2. The realization procedure for X, however, which was not an immediate one, required her to analyze the domain. At this point, 'J' was torn between the two realizations as the two realizations would take her to different conclusions about the continuity of the graph which resulted in a constant conflict in her utterances. This is a commognitive conflict between two of her own realizations for the signifier 'continuous function'.

DISCUSSION

The case study presented in the paper illustrates the rise of conflicts between different realizations for the same signifier when a student is confronted with an unfamiliar situation. This observation points to the need for an extension to the notion of "commognitive conflict" to encompass the conflicts between realizations for the same

signifier of the *same* individual. I have also attempted to frame these conflicting realizations as arising from nothing but the inconsistent definitions used for continuity; a concern that is seen to be present in textbooks, mathematical websites, or even arguably within classroom instruction and discourse.

As discussed, while there are inconsistencies in the way continuity of a function at a point is defined there is both ambiguity and inconsistency in explaining, let alone defining, what ‘a continuous function’ is.

In conclusion, I believe, apart from being aware of these problems that exist in electronic as well as in print resources that teachers and learners should have a clear picture of the issue and its roots so that they will at least be able to deal with ‘continuity’ problems according to the particular chosen definition. The study also gives evidence to the problematic situations that students are led to due to implied definitions that are not explicitly stated or taught. Hence, perhaps more importantly, what this study suggests in particular is that we also need to make a shift in our choices from a mathematical one to a pedagogical one when it comes to choosing definitions and making decisions about the kind of discourse we model in the classroom.

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